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Generation of fluid motion in a circular cylinder by an unsteady applied magnetic field

By ALFRED SNEYD†
Department of Applied Mathematics and Theoretical Physics,
Silver Street, Cambridge

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Three problems are considered. In the first, a uniform magnetic field is suddenly switched on outside an infinitely long circular cylinder of incompressible conducting fluid. As the field diffuses into the fluid, electric currents are generated and hence a Lorentz force field. Under the assumptions of small magnetic Reynolds number and small magnetic Prandtl number, the initial flow (before it has been modified by convection or viscous diffusion of vorticity) is calculated. In the second problem an initially uniform field is suddenly switched off outside a similar cylinder of fluid. It is shown that the switching off of the field produces a vorticity distribution identical with that produced by the switching on. The third problem considered is that of a circular cylinder of conducting fluid placed in a rapidly alternating magnetic field. It is shown that the alternating field produces vorticity at a constant rate, and a qualitative description of the resulting flow is given.

1. Introduction

The aim of this paper is to study situations in which a magnetic field is switched suddenly on or off around a body of conducting fluid. A magnetic field cannot penetrate conducting material instantaneously but diffuses in rather slowly (compared with the rate at which changes in electromagnetic fields are propagated in a vacuum, i.e. the speed of light), and as it diffuses in it must cause electric currents to flow in the conducting material and hence produce a Lorentz force field. Unless this Lorentz force field happens to be irrotational, in which case it will simply be balanced by a pressure gradient, it will produce some kind of fluid motion. Whether or not the force field is irrotational depends on the geometry of the container and the field that is switched on outside it. In the simplest possible case, that of a uniform magnetic field switched on parallel to the plane surface of a semi-infinite body of conducting fluid, it is easy to show that the force field is irrotational and consequently there is no motion. So it seems that if the fluid is to be moved, there must be some difference between the geometry of the field and the geometry of the container.

The simplest case of interest is then a uniform field switched on across an infinitely long circular cylinder of conducting fluid. It is assumed throughout that the fluid is incompressible and that the magnetic Reynolds number R_m is small

† Present address: University of Waikato, Hamilton, New Zealand.

(i.e. that the magnetic field is unaffected by the fluid flow). In § 2 a solution is obtained for the magnetic field. The assumption of small R_m means that the penetration time of the fluid by the magnetic field is so small that just after the field has diffused in the vorticity distribution produced has not yet had time to be modified significantly by convection. If the magnetic Prandtl number P_m is also assumed to be small it follows that this initial vorticity distribution will not be immediately modified by viscous diffusion. It then becomes a simple matter to calculate this initial vorticity distribution and the streamlines of the corresponding flow. This initial flow will later be modified by convection and diffusion of vorticity, but the solution of this non-linear problem is not attempted here.

If an initially uniform field is suddenly switched off outside a body of conducting fluid a similar effect occurs. The diffusion of the field out of the fluid will generally produce vorticity. In fact it is shown in § 3 that in the case of an infinitely long circular cylinder, the vorticity produced by the switching off of the field is identical with that produced by the switching on.

It should be pointed out that of course a magnetic field cannot be instantaneously switched on or off because of the self-inductance of the coils producing the field. The conclusions of §§ 2 and 3 will be valid only if the switching on (or off) time is small compared with the penetration time of the fluid by the field. In view of this, the situation considered in § 4, that of a circular cylinder of conducting fluid placed in an alternating field, is more realistic. The physical process driving this flow is essentially the same as in the switching on and off problems. The regular inward and outward diffusion of the field is a constant (in time) source of vorticity.

Situations have already been analyzed which involve flows generated by the same basic physical process that is evident here, namely the generation of Lorentz forces inside a body of conducting fluid by a time-varying magnetic field. Moffatt (1965) considered the problem of an infinitely long circular cylinder of conducting fluid placed in a rapidly rotating magnetic field, and showed that the fluid will rotate as a rigid body inside a viscous boundary layer. The methods of analysis used here are very similar to the methods used in Moffatt's paper.†

2. Field switched on across a circular cylinder

2.1. *Diffusion of the field*

Take cylindrical polar co-ordinates (r, θ, z) and suppose that the space $r \leqslant a$ is filled with fluid of density ρ, kinematic viscosity ν, and electrical conductivity σ. At time $t = 0$ a uniform magnetic field $B_0(\cos\theta, -\sin\theta, 0)$ is switched on at $r = \infty$. Let $\mathbf{B}_1(r, \theta, t)$ denote the magnetic field outside the cylinder, and $\mathbf{B}_2(r, \theta, t)$ the field inside. Since $\nabla . \mathbf{B}_1 = \nabla . \mathbf{B}_2 = 0$ it is possible to find stream functions ψ_1 and ψ_2 such that

$$\mathbf{B}_i = \left(\frac{1}{r}\frac{\partial \psi_i}{\partial \theta}, -\frac{\partial \psi_i}{\partial r}, 0\right) \quad (i = 1, 2).$$

† There is an error in equation (2.7) in Moffatt (1965) which affects equations (2.8) and (2.9) but none of the subsequent equations or discussion. The operator $\nabla^2 - 2/r^2$ in (2.7) should be replaced by $\nabla^2 - 1/r^2$, and the order of the Bessel functions in (2.8) and (2.9) should be 1, and not $\sqrt{2}$.

Just after the field has been switched on and before it begins to diffuse into the fluid, the field lines are exactly like the streamlines in the irrotational flow of an inviscid fluid around a solid cylinder. So the initial conditions are

$$\psi_1(r,\theta,0) = B_0(r-a^2/r)\sin\theta, \quad \psi_2(r,\theta,0) = 0. \tag{2.1}$$

It will be assumed that the magnetic Reynolds number of the flow is small, and under this assumption it is legitimate to neglect any effect associated with the fluid motion, and the equations satisfied by the magnetic field are

$$\nabla^2\psi_1 = 0, \quad \lambda\nabla^2\psi_2 = \partial\psi_2/\partial t, \tag{2.2a, b}$$

where $\lambda = 1/(\mu_0\sigma)$. The magnetic field must be continuous at the edge of the fluid cylinder and this provides the matching conditions

$$\left(\frac{\partial\psi_1}{\partial r}\right)_{r=a} = \left(\frac{\partial\psi_2}{\partial r}\right)_{r=a}, \quad \left(\frac{\partial\psi_1}{\partial\theta}\right)_{r=a} = \left(\frac{\partial\psi_2}{\partial\theta}\right)_{r=a}. \tag{2.3}$$

At large distances from the fluid cylinder the field must tend to become uniform, so

$$\psi_1 \to B_0 r\sin\theta \quad \text{as} \quad r\to\infty. \tag{2.4}$$

If one now writes:

$$\psi_1 = f_1(r,t)\sin\theta, \quad \psi_2 = f_2(r,t)\sin\theta,$$

the functions f_1 and f_2 must satisfy the following conditions:

$$f_1(r,0) = B_0(r-a^2/r), \quad f_2(r,0) = 0, \tag{2.1'}$$

$$\left.\begin{aligned} &\frac{\partial^2 f_1}{\partial r^2}+\frac{1}{r}\frac{\partial f_1}{\partial r}-\frac{1}{r^2}f_1 = 0,\\ &\lambda\left(\frac{\partial^2 f_2}{\partial r^2}+\frac{1}{r}\frac{\partial f_2}{\partial r}-\frac{1}{r^2}f_2\right) = \frac{\partial f_2}{\partial t},\end{aligned}\right\} \tag{2.2'}$$

$$f_1(a,t) = f_2(a,t), \quad \left(\frac{\partial f_1}{\partial r}\right)_{r=a} = \left(\frac{\partial f_2}{\partial r}\right)_{r=a}, \tag{2.3'}$$

$$f_1(r,t)\to B_0 r \quad \text{as} \quad r\to\infty. \tag{2.4'}$$

The first of equations (2.2′) and condition (2.4′) imply

$$f_1(r,t) = B_0[r-A(t)/r],$$

where $A(t)$ is an arbitrary function of time. Let $F_2(r,p)$ denote the Laplace transform (in t) of $f_2(r,t)$. The solution of the Laplace transform of the second of (2.2′) is

$$F_2(r,p) = B(p)\,J_1(i(p/\lambda)^{\frac{1}{2}}r),$$

where $B(p)$ is an arbitrary function of p. $A(t)$ and $B(p)$ may be calculated using the matching conditions (2.3′) with the final result

$$F_2(r,p) = \frac{2\beta_0 a J_1(i(p/\lambda)^{\frac{1}{2}}r)}{pi(p/\lambda)^{\frac{1}{2}}aJ_0(i(p/\lambda)^{\frac{1}{2}}a)}.$$

This Laplace transform may be inverted by contour integration to give

$$\psi_2(r,\theta,t) = B_0 r\sin\theta - 4B_0 a\sin\theta\sum_{n=1}^{\infty}\frac{J_1(\lambda_n r/a)}{\lambda_n^2 J_1(\lambda_n)}e^{-\lambda\lambda_n^2 t/a^2}, \tag{2.5}$$

where the λ_n are the positive zeros of $J_0(x)$. The Lorentz force field, $\mathbf{j}\times\mathbf{B}$, can now be calculated, and hence the fluid motion.

2.2. *Vorticity generated by the rotational Lorentz force*

The curl of the Navier–Stokes equation is

$$\partial\boldsymbol{\omega}/\partial t+(\mathbf{u}\,.\,\nabla)\,\boldsymbol{\omega} = (1/\rho)\,\nabla\times(\mathbf{j}\times\mathbf{B})+\nu\nabla^2\boldsymbol{\omega}, \tag{2.6}$$

where $\boldsymbol{\omega}(=\nabla\times\mathbf{u})$ is the vorticity. Now

$$\left|\frac{(\mathbf{u}\,.\,\nabla)\,\boldsymbol{\omega}}{\partial\boldsymbol{\omega}/\partial t}\right| = O(Ua/\lambda) = O(R_m),$$

where U is a typical value of the fluid speed. The time scale used here is a^2/λ which is the time scale for the inward diffusion of the magnetic field. The magnetic Reynolds number has already been assumed to be small, so the convective term in (2.6) may be neglected in comparison with the first term. Also

$$\left|\frac{\nu\nabla^2\boldsymbol{\omega}}{\partial\boldsymbol{\omega}/\partial t}\right| = O(\nu/\lambda)$$

and since $\nu/\lambda \ll 1$ for all liquid metals and electrolytes, the term $\nu\nabla^2\boldsymbol{\omega}$ in (2.6) will also be neglected. The physical meaning of these approximations is that the magnetic field diffuses into the fluid so quickly that the vorticity generated does not have time to be convected away or to diffuse away. With these approximations (2.6) takes the form

$$\partial\boldsymbol{\omega}/\partial t = (1/\rho)\,\nabla\times(\mathbf{j}\times\mathbf{B}). \tag{2.7}$$

This equation is valid only for a time of order a^2/λ after the magnetic field has been switched on. After the field has diffused in, the vorticity generated is convected and diffuses away, but the time scale for this process is much longer.

Integrating (2.7) with respect to t between the limits of 0 and ∞ and using the expression for $\mathbf{B}$ given by (2.5) gives (after a little algebra and summation of series by contour integration)

$$\omega(r,\infty) = \frac{8B_0^2\sin 2\theta}{\rho\mu_0\lambda}\left(\frac{a}{r}\right)\sum_{n=1}^{\infty}\frac{J_2(\lambda_n r/a)\,I_1(\lambda_n r/a)}{\lambda_n^2 J_1(\lambda_n)\,I_0(\lambda_n)}. \tag{2.8}$$

A graph of $\omega(r,\infty)$ as a function of r is shown in figure 1 (a). This represents the vorticity generated in the fluid just after the magnetic field has diffused in, and before the pattern has been modified by diffusion and convection.

Eventually this vorticity will be destroyed by viscosity, and also by the Lorentz forces due to the electric currents induced by the flow. The decay time due to viscosity is of order a^2/ν and the decay time due to the Lorentz forces is of order $\rho/\sigma B_0^2$.

2.3. *Stream function of the initial flow*

The next step is to calculate the fluid flow that corresponds to this distribution of vorticity. The flow is two-dimensional and the fluid is assumed to be incompressible, so there exists a stream function χ such that

$$u_r = (1/r)\,\partial\chi/\partial\theta, \quad u_\theta = -\partial\chi/\partial r.$$

Now,

$$\nabla^2\chi = -\omega_z = -f(r)\sin 2\theta,$$

where $f(r)$ is the coefficient of $\sin 2\theta$ given by (2.8). Writing

$$\chi = g(r) \sin 2\theta$$

gives the equation

$$\frac{d^2g}{dr^2} + \frac{1}{r}\frac{dg}{dr} - \frac{4}{r^2}g = f(r)$$

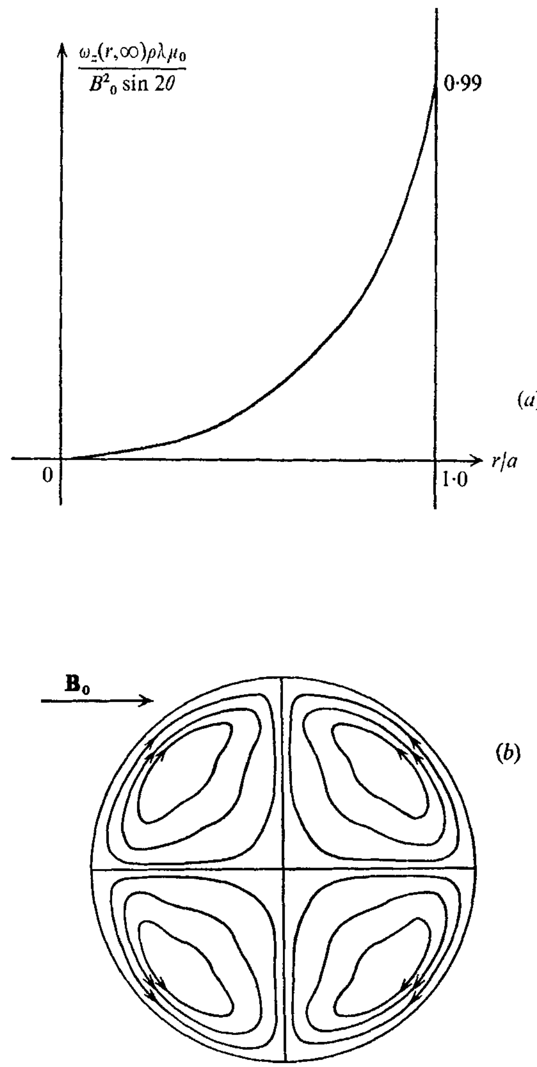

FIGURE 1

for determining g. The two boundary conditions on g are that $g(0)$ is finite and $g(a) = 0$ (i.e. that the edge of the fluid cylinder must be a streamline). The final solution is then:

$$\chi = \tfrac{1}{4}r^2 \sin 2\theta \left\{ \frac{1}{r^4}\int_0^r s^3 f(s)\, ds - \frac{1}{a^4}\int_0^a s^3 f(s)\, ds + \int_r^a \frac{1}{s} f(s)\, ds \right\}.$$

The streamlines for this initial flow are sketched in figure 1 (*b*). The main feature is that there are four eddies, one in each of the quadrants of the circular cross-section of the cylinder.

It is now possible to find the precise conditions for the validity of the low magnetic Reynolds number assumption. The fluid speed produced is of order $B_0^2 a/\mu_0 \lambda \rho$ so

$$R_m = B_0^2 a^2/\mu_0 \lambda^2 \rho.$$

For a cylinder of mercury of radius 10^{-1} m

$$R_m \ll 1 \Leftrightarrow B_0 \ll 1{\cdot}3 \text{ webers/m}^2 \text{ or } 13\,000 \text{ gauss}.$$

Only if the field strength is of order 1 weber/m² or greater need the further distortion of the field by the fluid be taken into account. For an electrolyte, the condition $R_m \ll 1$ would invariably be satisfied in situations of practical interest.

3. The effect of switching off an initially uniform field

It is now interesting to look at the way in which the magnetic field diffuses out of the fluid again when the external magnetic field is suddenly switched off. Let $\psi_1^*(r,\theta,t)$ and $\psi_2^*(r,\theta,t)$ be the stream functions for the magnetic field outside and inside the cylinder. Just after the field has been switched off and before it begins to diffuse out, there will be a uniform field with stream function $B_0 r \sin\theta$ inside the cylinder and a dipole field with stream function $B_0(a^2/r)\sin\theta$ outside. ψ_1^* and ψ_2^* must then satisfy the following:

$$\psi_1^*(r,\theta,0) = B_0(a^2/r)\sin\theta, \quad \psi_2^*(r,\theta,0) = B_0 r\sin\theta, \tag{3.1}$$

$$\nabla^2\psi_1^* = 0, \quad \lambda\nabla^2\psi_2^* = \partial\psi_2^*/\partial t, \tag{3.2}$$

$$\left(\frac{\partial\psi_1^*}{\partial\theta}\right)_{r=a} = \left(\frac{\partial\psi_2^*}{\partial\theta}\right)_{r=a}, \quad \left(\frac{\partial\psi_1^*}{\partial r}\right)_{r=a} = \left(\frac{\partial\psi_2^*}{\partial r}\right)_{r=a}, \tag{3.3}$$

$$\psi_1^* \to 0 \quad \text{as} \quad r \to \infty. \tag{3.4}$$

When one compares these equations with (2.1), (2.2), (2.3) and (2.4) it is clear that the solution for ψ_1^* and ψ_2^* is

$$\psi_1^* = B_0 r\sin\theta - \psi_1, \quad \psi_2^* = B_0 r\sin\theta - \psi_2.$$

Now it is possible to show that the diffusing out of the field produces exactly the same vorticity distribution as the diffusing in. The magnetic field may be divided into two parts, a uniform field $\mathbf{B}_0$, and a time dependent part $\mathbf{b}$ with stream function

$$-4aB_0\sin\theta \sum_{n=1}^{\infty} \frac{J_1(\lambda_n r/a)}{\lambda_n^2 J_1(\lambda_n)} e^{-\lambda\lambda_n^2 t/a^2}.$$

The field diffusing in is given by

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$$

and diffusing out it is given by

$$\mathbf{B} = -\mathbf{b}.$$

As the field diffuses in, the current density is $\mathbf{j}(=(1/\mu_0)\nabla\times\mathbf{b})$ which produces a vorticity field

$$\frac{1}{\rho}\int_0^\infty \nabla\times\{\mathbf{j}\times(\mathbf{B}_0+\mathbf{b})\}\,dt,$$

and as the field diffuses out, the current is $-\mathbf{j}$, producing a vorticity field

$$\frac{1}{\rho}\int_0^\infty \nabla\times(\mathbf{j}\times\mathbf{b})\,dt.$$

To prove these expressions are equal one must show that

$$\int_0^\infty \nabla\times(\mathbf{j}\times\mathbf{B}_0)\,dt = 0.$$

Since $\mathbf{B}_0$ is independent of t, and both $\mathbf{B}_0$ and $\int_0^\infty \mathbf{j}\,dt$ are solenoidal,

$$\int_0^\infty \nabla\times(\mathbf{j}\times\mathbf{B}_0)\,dt = \nabla\times\left(\int_0^\infty \mathbf{j}\,dt\times\mathbf{B}_0\right) = (\mathbf{B}_0.\nabla)\int_0^\infty \mathbf{j}\,dt.$$

$\mathbf{j}$ has only a z component and

$$j_z = -\frac{1}{\mu_0}\nabla^2\psi_2.$$

From (2.2*b*)

$$\partial\psi_2/\partial t = \lambda\nabla^2\psi_2,$$

and integrating this equation with respect to t between the limits of 0 and ∞ gives

$$\psi_2(r,\theta,\infty)-\psi_2(r,\theta,0) = -\mu_0\lambda\int_0^\infty j_z\,dt.$$

Thus

$$\int_0^\infty j_z\,dt = -\frac{1}{\mu_0\lambda}B_0 r\sin\theta = -\frac{B_0 y}{\mu_0\lambda}$$

and

$$(B_0.\nabla)\int_0^\infty \mathbf{j}\,dt = (\mathbf{B}_0.\nabla)\left(-\frac{B_0 y}{\mu_0\lambda}\right)\hat{\mathbf{k}} = 0.$$

If then a magnetic field is switched on outside the cylinder and left switched on for a time long enough for the field to diffuse into the fluid but not long enough for the vorticity generated to be convected or to diffuse away (i.e. for a time large compared with a^2/λ but small compared with $a^2/\lambda R_m$ or $a^2/\lambda P$) and then switched off again, the flow produced will be the same as that described by figure 1 except that the fluid speeds are doubled. This process of switching the field on and off may be repeated a number of times and each time this flow will be increased in strength. This may be continued until convection and diffusion of vorticity begin to modify the flow pattern and an eventual steady state will be reached.

To get an idea of the speed that can be generated in the fluid in this way, suppose that the magnetic field is left switched on for a time of order a^2/λ. A steady state will be reached when the average rate of generation of vorticity is balanced by the viscous diffusion of vorticity, i.e. when

$$\frac{B_0^2}{\rho\mu_0 a^2} = O\left(\frac{\nu U}{a^3}\right) \quad \text{or} \quad U = O\left(\frac{B_0^2 a}{\rho\mu_0\nu}\right).$$

For mercury in a pipe of radius 10^{-1} m and a magnetic field of 10^{-1} webers/m^2 this speed is 10^5 m/sec. For hydrochloric acid solution in the same pipe and with the same field strength this speed is again 10^5 m/sec. In each case of course the flow would become turbulent long before these speeds were reached, and the condition $R_m \ll 1$ would be violated in the case of mercury.

4. Circular cylinder in an alternating field

We now turn to the case in which an alternating (sinusoidal) field of fixed direction is applied outside a circular cylinder of conducting fluid. It will be assumed that the frequency ω of the field is large compared with λ/a^2 so that the magnetic field will be confined to a thin layer around the edge of the cylinder.

Take the same co-ordinate system as before, and suppose that at large distances from the cylinder, the magnetic field is of the form:

$$\mathscr{R}\{B_0(\cos\theta,\ -\sin\theta)\, e^{i\omega t}\}. \tag{4.1}$$

It is then natural to assume that the stream functions for $\mathbf{B}$ outside and inside the fluid, $\psi_1(r,\theta,t)$ and $\psi_2(r,\theta,t)$ take the form

$$\psi_1 = \mathscr{R}\{B_0 f_1(r)\sin\theta\, e^{i\omega t}\}, \quad \psi_2 = \mathscr{R}\{B_0 f_2(r)\sin\theta\, e^{i\omega t}\},$$

where f_1 and f_2 may be complex. In view of the first of (2.2′) and (4.1),

$$f_1(r) = r + B/r,$$

where B is an arbitrary constant. The second of (2.2′) implies that

$$f_2(r) = CJ_1(\alpha r),$$

where $\alpha^2 = -i\omega/\lambda$ and C is a constant. The boundary conditions (2.3′) can be used to evaluate the constants B and C with the final result

$$\psi_2 = \frac{2B_0}{\alpha}\left(\frac{a}{r}\right)^{\frac{1}{2}}\frac{J_1(\alpha r)}{J_0(\alpha a)}\sin\theta\, e^{i\omega t}. \tag{4.2}$$

At this point it is convenient to use the assumption that the field is alternating very rapidly, i.e.

$$a^2/\lambda \gg 1/\omega \quad \text{or} \quad |\alpha a| \gg 1.$$

It is now possible to obtain the following approximate expression for ψ_2 using the asymptotic expansion for Bessel functions of large arguments.

$$\psi_2 = \frac{2^{\frac{1}{2}}}{\beta}B_0\left(\frac{a}{r}\right)^{\frac{1}{2}} e^{-\beta(a-r)}\cos\xi\sin\theta, \tag{4.3}$$

where $\beta = (\omega/2\lambda)^{\frac{1}{2}}$ and $\xi = \beta(r-a)+\omega t - \frac{1}{4}\pi$.

The curl of the Lorentz force can now be calculated and

$$[\nabla\times(\mathbf{j}\times\mathbf{B})]_z = \frac{2B_0^2\beta}{\mu_0 a}e^{-2\beta(a-r)}\sin 2\theta. \tag{4.4}$$

(This equation is almost identical with equation (2.15) of Moffatt (1965).) It is interesting that this last expression is independent of ξ, and hence of t. This means that vorticity is being generated at a constant rate in a thin layer (of thickness of order β^{-1}) around the surface of the cylinder. This vorticity will diffuse into the fluid and eventually set up some kind of steady flow pattern. In the calculation of this flow pattern, there are two separate cases to consider, that of low Reynolds number and that of high Reynolds number.

Case I: *Low Reynolds number*

In this case the equation of vorticity takes the linear form

$$(1/\rho)\nabla\times(\mathbf{j}\times\mathbf{B})+\nu\nabla^2\boldsymbol{\omega}=0.$$

In terms of the stream function χ for $\mathbf{u}$ this may be written as

$$\nabla^4\chi=\frac{2B_0^2\beta}{\mu_0 a\rho\nu}e^{-2\beta(a-r)}\sin 2\theta.$$

The right-hand side of this equation is very small except in a thin layer around the edge of the cylinder and, in this layer, $\partial/\partial r \gg \partial/\partial\theta$ so the equation may be approximated by

$$\frac{\partial^4\chi}{\partial r^4}=\frac{2B_0^2\beta}{\mu_0 a\rho\nu}e^{-2\beta(a-r)}\sin 2\theta.$$

The solution of this equation that satisfies the two boundary conditions: $(\chi)_{r=a}=0$ (that the pipe wall must be a streamline) and $(\partial\chi/\partial r)_{r=a}=0$ (the no-slip condition at the pipe wall) is

$$\chi=\frac{B_0^2}{8\mu_0 a\rho\nu\beta^3}\left\{e^{-2\beta(a-r)}-\frac{\beta r^4}{a^3}+\frac{\beta r^2}{a}\right\}\sin 2\theta.$$

Again, the most important feature of the streamline pattern is that there are four eddies in each of the four quadrants of the circular cross-section of the cylinder.

The assumption $R \ll 1$ requires that, for example, for a pipe of mercury of radius 10^{-1} m, the flow speeds must be much less that 10^{-6} m/sec, and for the same pipe of hydrochloric acid the speeds must be much less than 10^{-5} m/sec. So it seems more interesting and realistic to consider the case of high Reynolds number.

Case II: *High Reynolds number*

It is more difficult to determine the flow in this case since the equation of motion is non-linear. Perhaps the best way to get a picture of the flow is to trace its development from the moment of switching-on of the alternating field. Figure 2 describes schematically the flow produced in a cylinder of conducting fluid when a magnetic field has been switched on outside the cylinder for a time which is short compared with a^2/λ (the penetration-time of the cylinder by the magnetic field). Figure 2 (*a*) shows the vorticity distribution, which can be calculated from the formulae in § 2. The vorticity is effectively confined to the layer along the edge of the cylinder into which the magnetic field has had time to diffuse. Figure 2 (*b*) shows the streamlines that correspond to this distribution of vorticity.

Outside the magnetic penetration layer, there is no vorticity and the velocity stream function is proportional to $r^2 \sin 2\theta$. If the applied field continues to oscillate rapidly, one would expect the streamline pattern to maintain the same general shape, while the flow speeds increase. Eventually, when a steady state is

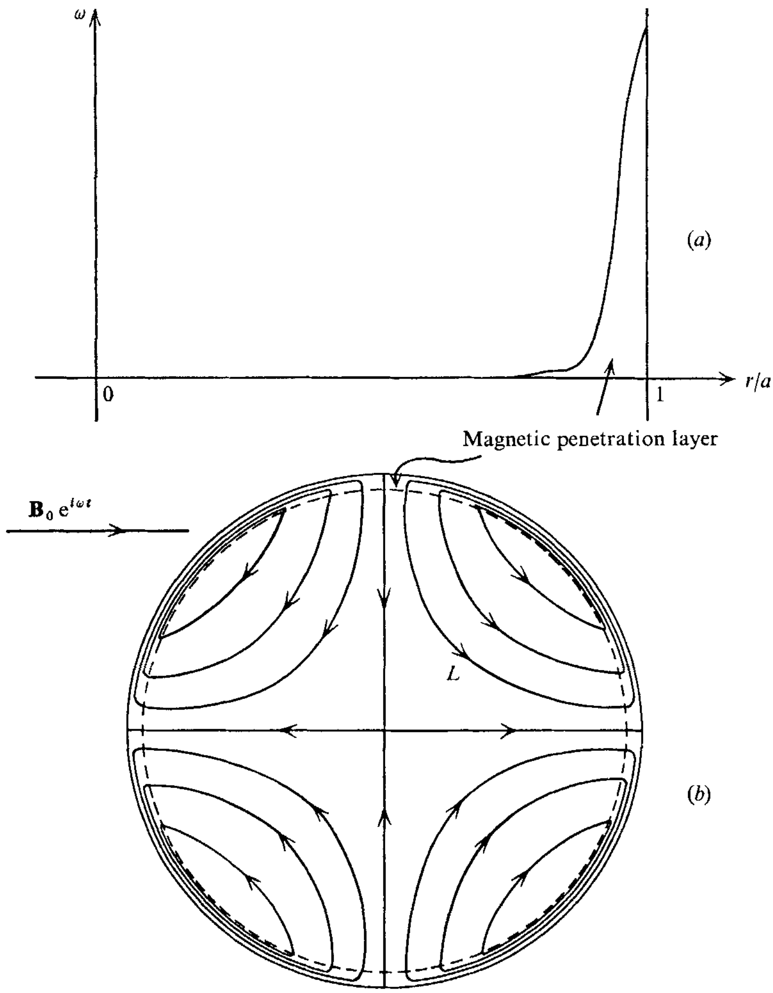

FIGURE 2

reached, some vorticity must have diffused and been convected from the magnetic penetration layer into the interior of the cylinder, and the streamlines will then be somewhat different in shape from figure 2(*b*). It is also possible that some streamlines may no longer pass through the boundary layer. If such a region of closed streamline forms in the interior of the cylinder, it follows from a theorem of Batchelor (1956) that the vorticity there must be uniform.

In any case it is possible to obtain an estimate of the flow speeds generated in the cylinder. The Navier–Stokes equation for steady flow may be written in the form

$$\boldsymbol{\omega} \times \mathbf{u} = -\nabla(\tfrac{1}{2}u^2 + p/\rho) + (1/\rho)\,\mathbf{j} \times \mathbf{B} + \nu\nabla^2\mathbf{u}.$$

If L (figure 2 (*b*)) is a streamline passing through the magnetic penetration layer, then the line integral of this equation around L gives

$$\frac{2B_0^2\beta}{\mu_0 a}\int_S e^{-2\beta(a-r)}\sin 2\theta\, dS+\rho\nu\int_L \nabla^2\mathbf{u}\,.\,\mathbf{dl} = 0, \tag{4.5}$$

where S is the area enclosed by L. Replacing each term of this last equation by an order of magnitude gives

$$B_0^2/\mu_0 = O(\rho\nu\beta^2 aU) \quad \text{and} \quad U = O(\beta_0^2/\mu_0 a\rho\nu\beta^2).$$

It is interesting that this order of magnitude is the same as that for the low Reynolds number case. This is to be expected since it is the magnetic boundary layer which drives the flow, and in this layer there is essentially a balance between viscous and magnetic forces, which is demonstrated in (4.5).

If $\beta a = 10^n$, then for a pipe of mercury of radius 10^{-1} m and a field strength of 10^{-1} webers/m^2, a typical flow speed is 10^{7-2n} m/sec. The assumption $\omega \gg \lambda/a^2$ implies that $n \geqslant 1$. $n = 1$ gives a flow speed of 10^5 m/sec, but of course the flow would become turbulent and the condition $R_m \ll 1$ would be violated before such a speed could be attained.

5. Conclusion

It has been shown that switching magnetic fields on and off outside a pipe of conducting fluid, or placing the pipe in an alternating field, will produce motion in the fluid in the form of four cylindrical eddies, one in each quadrant of the cross-section of the pipe. The flow speeds that are produced in this way are large enough to be observed experimentally, using either mercury or an electrolyte as the conducting fluid, so this should provide a good opportunity of comparing magnetohydrodynamic theory and experiment. Of course it is not possible to switch on a magnetic field instantaneously (because of the self-inductance of the coils) but provided the switching-on time is much smaller than the time for convection of vorticity (a/U_0) or viscous diffusion of vorticity (a^2/ν) the flow patterns should not be too different from those described in figure 1.

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